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# Dynamic Analysis of Harmonically Excited Non-Linear System Using Multiple Scales Method

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An analytical method is presented for evaluation of the steady state periodic behavior of nonlinear systems. This method is based on the substructure synthesis formulation and a MS (multiple scales) procedure, which is applied to the analysis of nonlinear responses. The proposed procedure reduces the size of large degrees-of-freedom problem in solving nonlinear equations. Feasibility and advantages of the proposed method are illustrated with the nonlinear rotating machine system as an example of large mechanical structure systems. In addition, its efficiency for nonlinear response prediction will be shown by comparison of other conventional methods.

**Key Words :** Nonlinear Mechanical System, Response Analysis, Modal Analysis, Multiple Scales Method, Multi-DOF System, Modeling of Complex System, Dynamic Design of System

## 1. Introduction

In recent years, industrial machines used for the gas turbine for propulsion of an aircraft, power plant turbine etc. trend toward high-speed and lightweightdur, which may cause the trouble of nonlinear vibration. Vibration analysis of such rotor systems is performed usually by the FEM (Finite Element Method) with linear model. When large amplitude vibration occurs, however, nonlinear characteristics of the rotor systems with complexity can not be represented simply with linear spring and docmping coefficients. Therefore, it is necessary to investigate the nonlinear characteristics in vibration analysis and design of rotor systems. On the other hand, a high-speed rotor system used for the gas turbine for propulsion of an aircraft, power plant turbine etc. promptly pass a critical speed. Accordingly, the casing is often modeled elastically to decrease the critical speed. Vibration induced in the rotor-bearing-casing system may make the casing contact the rotor and then give rise to damages of the bearing possibly. Therefore, the investigation of the response of a rotating machine is very important for stable operation. To construct real mathematical model in vibration analysis, dynamic characteristics of rotor, bearing and casing should be investigated.

For efficient Vibration analysis of a mechanical system with a large number of DOF's, the SSM (Substructure synthesis method) has been studied. Iwatsubo et al. (1998) and Moon et al. (1999; 2001) presented analytical methods to analyze the vibration of a nonlinear rotor-bearing-casing system by employing the perturbation method. They considered the nonlinearity in the shaft and bearing part and considered the effect of nonlinear sensitivity in the subsystem. Moon et al. (2001) proposed an approximate analytical method to analyze the dynamic problems of a nonlinear structure system using the SSM and a harmonic balance method.

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However, a nonlinear vibration problem needs more accurate analysis in some rotor systems, which are used in jet engine of an aircraft or some power plant turbines. The high-speed and light-weight mechanical systems lead to more complex nonlinear vibration. In the analysis of nonlinear system, there have been a lot of research works using the method of MS for the single DOF of nonlinear vibration system, and its application to the multi DOF system have been reported (Haquang et al., 1987; 1978). However, the study, which applied the on the MS method which is applied to the nonlinear vibration analysis of rotor system, has not been reported yet.

Therefore, this paper presents an analytical technique based on the MS theory and the mode superposition principle for the dynamic analysis of nonlinear mechanical systems. Furthermore, the proposed method enhanced the previous studies (Iwatsubo, et al., 1998; Moon, et al., 1999; 2001) such that it can be applied to more accurate analysis comparing with the perturbation method of the previous studies. The proposed method is then applied to a nonlinear mechanical system in order to illustrate the performance of the method in respect of the computational accuracy by comparing the results obtained from the other conventional methods.

## 2. Method of Analysis

A structural system consists of a set of inter-connected components that have segments with distributed mass elasticity and nonlinear parts. The first stage in analysis process, therefore, is to sub-structure the original nonlinear system into some components that can be modeled separately with linear and nonlinear sets. Small substructures may be easier to model and will eventually result in an economical analysis procedure.

When a complex large system is modeled with the SSM, the internal force is considered because each component can be synthesized through the internal force with the other components as shown in Fig. 1. In this case the internal forces act on the component 1 (1b) and component 2(2b) in the opposite direction with the same magnitude. Those internal forces will disappear by synthesizing each component into the overall system. In this study the external force (1a), which acts on the rotor, is unbalance force.

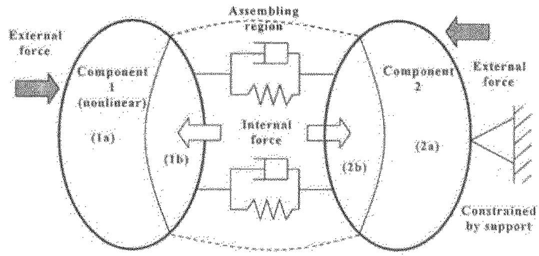


**Fig. 1** Sub-structuring of the complex system

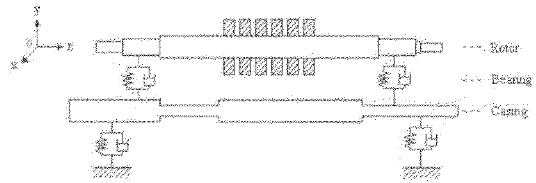


**Fig. 2** Rotor-bearing-casing system

### 2.1 Modeling of the nonlinear system

In this paper, we consider a rotor-bearing-casing system as shown in Fig. 2. The rotor is supported by bearings that are fixed on the casing. The casing and the foundation are elastically connected. The rotor has material nonlinearity. For dynamic analysis of this kind of complex system, the SSM can be applied.

The coordinates system of the rotor-bearing-casing system is defined as shown in Fig. 2. The $o$-$xyz$ coordinate is fixed in the reference frame, where the $x$-axis is perpendicular to directions of shaft and casing, the $y$-axis vertically upwards, and the $z$-axis along the shaft and the casing. The shaft and casing components are modeled by using the FEM. The rotor system is assumed to be excited by the unbalance force. Then assumption of a steady state response is reasonable. The excitation forces by the eccentric mass $m_i$ at a distance $e_i$ from the rotor geometric center are given by

$$\{{}^1F_u(\Omega, t)\} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{Bmatrix} m_i e_i \Omega^2 \cos(\Omega t + \phi_i) \\ m_i e_i \Omega^2 \sin(\Omega t + \phi_i) \end{Bmatrix}, \quad (1)$$

$$(i = 1, 2, 3, \cdots)$$

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where $\phi$ and $\Omega$ are the phase quantity and the rotating frequency, respectively. The excitation force by the eccentric mass of the rotor can be treated as a harmonic excitation. In general, the response shows well the nonlinear characteristics near natural frequencies of in the nonlinear system, as observed in the single DOF system. Espe-ially in the rotor system, the dynamic behavior around the critical speed is very important where most of the troubles occur. Therefore, it needs to pass the critical speed quickly without troubles. Because of these reasons, the exciting frequency around the first natural frequency of the system should taken into account.

### 2.2 Modeling of nonlinear component

When the rotor is modeled by the FEM, the nonlinear characteristic of the restoring force is regarded as nonlinear-displacement-dependent-stiffness. By considering the boundary conditions, the equation of motion for the nonlinear component can be written as (Moon et al., 1999; 2001)

$$[{}^{1}M]\{{}^{1}\ddot{u}\}+[{}^{1}K]\{{}^{1}u\}+\varepsilon[K_N]\{{}^{1}u^{3}\} \\ =\{{}^{1}F_u(\Omega,\ t)\}+\{{}^{1}F_b\} \tag{2}$$

where $[{}^{1}M]$, $[{}^{1}K]$ and $[K_N]\{{}^{1}u^{3}\}$ are the mass, stiffness matrices and nonlinear term, respectively. $\{{}^{1}F_u(t)\}$, $\{{}^{1}F_b\}$ are an external unbalance force vector by the rotor and an internal force vector, respectively. $\varepsilon$ is a small parameter. The superscript denotes the nonlinear component. The displacement vector can be written as

$$\{{}^{1}u\}=\{x_i,\ \theta_{xi},\ y_i,\ \theta_{yi}\}^{T},\ (i=1,\ 2,\ \cdots,\ n) \tag{3}$$

where $x_i$, $\theta_{xi}$, $y_i$ and $\theta_{yi}$ are the displacements and rotations in the $x$-direction and $y$-direction at the $i$-th nodal point, and $n$ the number of nodes. Exactly to say, vibration modes of a nonlinear system are slightly different from those of a linear system. But for simplicity of analysis, they are assumed to keep those of a linear one. Accordingly, the modal coordinate system can be obtained using the modal matrix $[{}^{1}\Phi]$. Then, the Eq. (2) can be transformed into the modal coordinate $\{{}^{1}\xi\}$ system as follows:

$$\{{}^{1}u\}=[{}^{1}\Phi]\{{}^{1}\xi\}, \tag{4}$$

$$\{{}^{1}\ddot{\xi}\}+[{}^{\backslash 1}\omega^{2}_{\backslash}]\{{}^{1}\xi\}+\varepsilon[{}^{1}\Phi]^{T}[K_N]\{{}^{1}u^{3}\} \\ =\varepsilon\{{}^{1}f_u(\Omega,\ t)\}+\{{}^{1}f_b\} \tag{5}$$

where, $\{{}^{1}f_u\}(=[{}^{1}\Phi]^{T}\{{}^{1}F_u(t)\})$, $\{{}^{1}f_b(t)\}(=[{}^{1}\Phi]^{T}\{{}^{1}F_b\})$ are the external and internal forces in modal coordinates, respectively. Since $[{}^{1}\Phi]^{T}[K_N]\{{}^{1}u^{3}\}$ is not a diagonal matrix, this term is changed into modal coordinates in accordance with the reasonable procedure (Moon et al., 1999; 2001). Then, a nonlinear term can be derived as $\varepsilon[{}^{\backslash}k'_{N\backslash}]\{\xi_1^{3}\}$.

Here, the perturbation method is introduced to solve the nonlinear equation in Eq. (5). The variant $\varepsilon[{}^{\backslash 1}k_{N\backslash}]$ can be regarded as the perturbation parameter term, because it is relatively smaller than $[{}^{\backslash 1}\omega^{2}_{\backslash}]$. Thus $\{{}^{1}\xi\}$ can be expanded in terms of $\varepsilon$ follows;

$$\{{}^{1}\xi\}=\{{}^{1}\xi^{(0)}\}+\varepsilon\{{}^{1}\xi^{(1)}\}+\varepsilon^{2}\{{}^{1}\xi^{(2)}\}+\cdots \tag{6}$$

where the superscript $(\cdot)$ denotes the perturbation order. By substituting Eq. (6) into Eq. (5), and arranging by $\varepsilon$, the perturbed equations can be written as

$$\begin{aligned} &\{{}^{1}\ddot{\xi}^{(0)}\}+[{}^{\backslash 1}\omega^{2}_{\backslash}]\{{}^{1}\xi^{(0)}\}=\{{}^{1}f_b{}^{(0)}\}, \\ &\{{}^{1}\ddot{\xi}^{(1)}\}+[{}^{\backslash 1}\omega^{2}_{\backslash}]\{{}^{1}\xi^{(1)}\} \\ &=\{{}^{1}f_u\}+\{{}^{1}f_{p1}({}^{1}\xi^{(0)})\}+\{{}^{1}f_b{}^{(1)}\}, \\ &\{{}^{1}\ddot{\xi}^{(2)}\}+[{}^{\backslash 1}\omega^{2}_{\backslash}]\{{}^{1}\xi^{(2)}\} \\ &=\{{}^{1}f_{p2}({}^{1}\xi^{(0)2},\ {}^{1}\xi^{(1)})\}+\{{}^{1}f_{p1}({}^{1}\xi^{(0)})\}+\{{}^{1}f_b{}^{(2)}\} \end{aligned} \tag{7}$$

where $\{{}^{1}f_b{}^{(0)}\}$, $\{{}^{1}f_b{}^{(1)}\}$ and $\{{}^{1}f_b{}^{(2)}\}$ are perturbed internal forces. $\{{}^{1}f_{p1}\}$ and $\{{}^{1}f_{p2}\}$ include the nonlinear stiffness term, $\{{}^{1}f_{p1}({}^{1}\xi^{(0)})\}=-[{}^{\backslash 1}k_{\backslash}]\{{}^{1}\xi^{(0)3}\}$, $\{{}^{1}f_{p2}({}^{1}\xi^{(0)2},\ {}^{1}\xi^{(1)})\}=\{-3[{}^{\backslash 1}k_{\backslash}]\{{}^{1}\xi^{(0)2}\cdot{}^{1}\xi^{(1)}\}\}$, respectively. Here $\{{}^{1}\xi^{(0)2}\cdot{}^{1}\xi^{(1)}\}$ is a perturbed modal displacement term which comes from the perturbation zero-th order and perturbation first order.

### 2.3 Modeling of linear, assembling components and overall system

The casing is modeled as a linear substructure. With the eigenvalue analysis, the equation of motion in the modal coordinates can be obtained as follows;

$$[{}^{\backslash}I_{\backslash}]\{{}^{2}\ddot{\xi}\}+[{}^{\backslash 2}\omega^{2}_{\backslash}]\{{}^{2}\xi\}=\varepsilon\{f_c\}+\{{}^{1}f_b\} \tag{8}$$

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where $[{}^{\backslash 2}\omega_i^2]$ and $[{}^{\backslash}I_{\backslash}]$ are eigenvalue of linear substructure and identity matrix, respectively, and $[f_c]$ the external force vector. Even though the casing component is linear system, this component is perturbed the same as the nonlinear component, because the higher order harmonic oscillation which occurs in the nonlinear component is translated through the higher order perturbed equation in Eq. (9)

$$\begin{aligned}
&\{{}^2\ddot{\xi}^{(0)}\}+[{}^{\backslash 2}\omega_i^2]\{{}^2\xi^{(0)}\}=-\{{}^2f_b^{(0)}\},\\
&\{{}^2\ddot{\xi}^{(1)}\}+[{}^{\backslash 2}\omega_i^2]\{{}^2\xi^{(1)}\}=-\{f_c\}-\{{}^2f_b^{(1)}\},\\
&\{{}^2\ddot{\xi}^{(2)}\}+[{}^{\backslash 2}\omega_i^2]\{{}^2\xi^{(2)}\}=-\{{}^2f_b^{(2)}\}
\end{aligned} \quad (9)$$

where $\{{}^2f_b^{(0)}\}$, $\{{}^2f_b^{(1)}\}$ and $\{{}^2f_b^{(2)}\}$ are the perturbed internal forces.

Ball bearings are considered. As an assembling component, Damping in the bearing, is ignored in this study for the simplicity of bearing model in order to verify the effect of nonlinear restoring force. The restoring force of the bearing is modeled as

$$[{}^1k_{b1}]\{{}^1x_b\}=\{{}^1f_b\},\ [{}^2k_{b2}]\{{}^2x_b\}=-\{{}^2f_b\} \quad (10)$$

where $[{}^jk_{bj}]\,(j=1,\ 2)$ are bearing coefficient, $\{{}^1f_b\}$, $\{{}^2f_b\}$ the internal force vectors of the nonlinear component and linear component, respectively, and $\{{}^jx_b\}$ the relative displacements between the rotor and casing corresponding to the bearings. In order to solve the overall equation, the small parameter is set equal to the perturbation parameter of the nonlinear component. The internal force vectors can be perturbed as

$$\{{}^1f_b\}=\{{}^1f_b^{(0)}\}+\varepsilon\cdot\{{}^1f_b^{(1)}\}+\varepsilon^2\cdot\{{}^1f_b^{(2)}\} \quad (11)$$

$$\{{}^2f_b\}=\{{}^2f_b^{(0)}\}+\varepsilon\cdot\{{}^2f_b^{(1)}\}+\varepsilon^2\cdot\{{}^2f_b^{(2)}\} \quad (12)$$

In order to synthesize the components, Eqs. (7), (9) and (12) are combined and rewritten according to the equation of order $\varepsilon^{(p)}\,(p=0,1,2)$

$$\{\ddot{\xi}^{(p)}\}+[\bar{K}^{(p)}]\{\xi^{(p)}\}=\{F^{(p)}(\Omega,t,\xi^{(0)},\xi^{(1)})\} \quad (13)$$

where $[\bar{K}^{(p)}]$ is the stiffness matrix of the overall system

$$\begin{aligned}
&\{\xi^{(0)}\}=\{\{{}^1\xi^{(0)}\}^T,\{{}^1x_b^{(0)}\}^T,\{{}^2x_b^{(0)}\}^T,\{{}^2\xi^{(0)}\}^T\},\\
&\{F^{(0)}\}=\{\{0\}^T,\{-{}^1f_b^{(0)}\}^T,\{{}^2f_b^{(0)}\}^T,\{0\}^T\},\\
&\{\xi^{(1)}\}=\{\{{}^1\xi^{(1)}\}^T,\{{}^1x_b^{(1)}\}^T,\{{}^2x_b^{(1)}\}^T,\{{}^2\xi^{(1)}\}^T\},\\
&\{F^{(1)}\}=\{\{{}^1f_u\}^T,\{{}^1f_{p1}\}^T,\{-{}^1f_b^{(0)}\}^T,\{{}^2f_b^{(1)}\}^T,\{f_c^{(0)}\}^T\},\\
&\{\xi^{(2)}\}=\{\{{}^1\xi^{(1)}\}^T,\{{}^1x_b^{(1)}\}^T,\{{}^2x_b^{(1)}\}^T,\{{}^2\xi^{(1)}\}^T\},\\
&\{F^{(2)}\}=\{\{{}^1f_{p2}\}^T,\{-{}^1f_b^{(1)}\}^T,\{{}^2f_b^{(1)}\}^T,\{0\}^T,\{0\}^T\}
\end{aligned}$$

In order to apply the SSM, we introduce the transformation matrix, which is composed of $[\phi_{bi}]\,(i=1,\ 2)$, and the eigenvector matrix of the assembling region. (Moon, et al., 1999, 2001) By substituting the transformation matrix into Eq. (13) and pre-multiplying, the overall equation of order $\varepsilon^{(p)}$ can be expressed as

$$\begin{Bmatrix}{}^1\ddot{\xi}_i^{(p)}\\{}^2\ddot{\xi}_i^{(p)}\end{Bmatrix}+\begin{bmatrix}[{}^1\omega_i^2]+[a_1] & [a_2]\\ [a_3] & [{}^2\omega_i^2]+[a_4]\end{bmatrix}\begin{Bmatrix}{}^1\xi_i^{(p)}\\{}^2\xi_i^{(p)}\end{Bmatrix}=\begin{Bmatrix}{}^1f_\eta^{(p)}\\{}^2f_\eta^{(p)}\end{Bmatrix}=\{f_\eta^{(p)}\} \quad (14)$$

$$\begin{aligned}
&[a_1]=[\phi_{b1}]^T[{}^1k_{b1}][\phi_{b1}],\ [a_2]=[\phi_{b1}]^T[{}^2k_{b1}][\phi_{b2}],\\
&[a_3]=[\phi_{b2}]^T[{}^1k_{b2}][\phi_{b1}],\ [a_4]=[\phi_{b2}]^T[{}^2k_{b2}][\phi_{b2}].
\end{aligned}$$

The external force with order $\varepsilon^{(p)}$ is obtained as

$$\begin{aligned}
&\{f_\eta^{(0)}\}=\begin{Bmatrix}[\phi_{a1}]^T\cdot\{0\}^T\\ [\phi_{a2}]^T\cdot\{0\}^T\end{Bmatrix},\\
&\{f_\eta^{(1)}\}=\begin{Bmatrix}[\phi_{a1}]^T\cdot(\{{}^1f_{p1}\}^T+\{{}^1f_u\}^T)+[\phi_{b1}]^T\cdot\{{}^1f_b^{(1)}\}\\ [\phi_{a2}]^T\cdot\{{}^2f_c^{(0)}\}^T+[\phi_{b2}]^T\cdot\{{}^2f_b^{(1)}\}\end{Bmatrix},\\
&\{f_\eta^{(2)}\}=\begin{Bmatrix}[\phi_{a1}]^T\cdot\{{}^1f_{p2}\}^T+[\phi_{b1}]^T\cdot\{{}^1f_b^{(2)}\}\\ [\phi_{a2}]^T\cdot\{0\}+[\phi_{b2}]^T\cdot\{{}^2f_b^{(2)}\}\end{Bmatrix}.
\end{aligned}$$

By applying the modal analysis technique $\{\xi\}=[\Phi_z]\{\eta\}$, Eq. (14) can be solved, where $[\Phi_z]$ is the modal matrix of the overall structure.

$$\begin{aligned}
&\{\ddot{\eta}^{(0)}\}+[\backslash\omega_z^2\backslash]\{\eta^{(0)}\}=\{0\}\\
&\{\ddot{\eta}^{(1)}\}+[\backslash\omega_z^2\backslash]\{\eta^{(1)}\}=-[Q]\{\dot{\eta}^{(0)}\}+\{G\}-[P]\{\eta^{(0)3}\}\\
&\{\ddot{\eta}^{(2)}\}+[\backslash\omega_z^2\backslash]\{\eta^{(2)}\}=-[Q]\{\dot{\eta}^{(1)}\}-3[P]\{\eta^{(0)2}\cdot\eta^{(1)}\}
\end{aligned} \quad (15)$$

where $[Q]=[\Phi_Z][{}^{\backslash}C_{\backslash}][\Phi_Z]^T$, $\{G\}=[\Phi_Z]\{{}^1f_u\}$ and $[P]=[\Phi_Z][{}^{\backslash 1}k_{N\backslash}][\Phi_Z]$. $[\backslash\omega_z^2\backslash]$ is the eigenvalue of the overall system. Here $\{\eta^{(0)2}\cdot\eta^{(1)}\}$ is a perturbed modal displacement term which comes from the perturbation zero-th order and perturbation first order.

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In this study, the damping term is considered in the overall system as a proportional damping of $[\ C\ ]=\alpha[I]+\beta[\ \omega_z^2\ ]$ where $\alpha, \beta$ are the damping coefficients.

## 3. Response Analysis by Applying the MS Method

To obtain the equation to perturbation first order by MS method, time scale is introduced as follows;

$$\begin{aligned} &T_n=\varepsilon^n t, \\ &\frac{d}{dt}=\frac{dT_0}{dt}\frac{\partial}{\partial T_0}+\frac{dT_1}{dt}\frac{\partial}{\partial T_1}+\frac{dT_2}{dt}\frac{\partial}{\partial T_2}+\cdots \\ &\quad=D_0+\varepsilon D_1,+\varepsilon^2 D_2+\cdots, \\ &\frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0D_2)+\cdots \end{aligned} \tag{16}$$

By substituting Eq. (16)) into Eq. (15), and by arranging it with $\varepsilon$, the equations can be rewritten as

$$\begin{aligned} &D_0^2\{\eta^{(0)}\}+[\ \omega_z^2\ ]\{\eta^{(0)}\}=\{0\} \\ &D_0^2\{\eta^{(1)}\}+[\ \omega_z^2\ ]\{\eta^{(1)}\}=-D_0D_1\{\eta^{(0)}\}-2[Q]D_0\{\eta^{(0)}\} \\ &\qquad -[P]\{\eta^{(0)3}\}+\{G\} \\ &D_0^2\{\eta^{(2)}\}+[\ \omega_z^2\ ]\{\eta^{(2)}\}=-D_0D_1\{\eta^{(1)}\}-(D_1^2+2D_0D_2)\{\eta^{(0)}\} \\ &\qquad -2[Q]D_1\{\eta^{(0)}\}-2[Q]D_0\{\eta^{(1)}\} \\ &\qquad -[P]\{\eta^{(0)2}\cdot\eta^{(1)}\} \end{aligned} \tag{17}$$

The exciting frequency is regarded as a value near to the first natural frequency $\omega_1$. By noting the detuning parameter $\sigma$, the exciting frequency can be expressed as

$$\Omega=\omega_1+\varepsilon\sigma \tag{18}$$

Here, only the main resonance is considered by assuming that there is no other resonance except the main resonance.

The solution of the first equation Eq. (17) can be expressed as

$$\{\eta^{(0)}\}=\{A\}\exp(i\omega_0 T_0)+\{\bar{A}\}\exp(-i\omega_0 T_0) \tag{19}$$

According to the MS theory, by substituting Eq. (18), and Eq. (19) into Eq. (17), the equation can be expressed in the single DOF form for the first mode. The secular term is eliminated from the particular solution. In a similar way, a condition to eliminate the secular term of other components of equation for $m=2, 3, \sim 2n$ can be applied. When the vibration steady state, by dividing the equation into real part and imaginary part and then squaring each equation, the equation can be written as

$$(Q_{11}a_1)^2+\left(\sigma a_1-\frac{3}{8\omega_1}P_{11}a_1^3\right)^2=\frac{1}{4\omega_1^2}G_1^2, \tag{20}$$
$$2_m=0\ (m=2, 3, \sim 2n)$$

The frequency response of the system to the perturbation first order is obtained by solving Eq. (20). Next, a formulation procedure to obtain the equation to perturbation second order is introduced. According to the second equation of Eq. (17), the particular solution for the single DOF is obtained by eliminate the secular term.

$$\begin{aligned} \eta_1^{(1)}=&\frac{1}{\omega_k^2-\omega_1^2}\left(2i\sum_{k=2}^{n}\omega_k Q_{1k}A_k+3\sum_{k=2}^{n}P_{1k}A_k^2\bar{A}_k\right)\exp(i\omega_k T_0) \\ &+\frac{1}{9\omega_k^2-\omega_1^2}\sum_{k=1}^{n}P_{1k}A_k^3\exp(3i\omega_k T_0)+cc \end{aligned} \tag{21}$$

Similarly the particular solution of equation for $m=2\sim 2n$ is obtained by eliminate the secular term as

$$\begin{aligned} \eta_m^{(1)}=&\frac{1}{\omega_k^2-\omega_m^2}\left(2i\sum_{k=1}^{n}\omega_k Q_{mk}A_k+3\sum_{k=1}^{n}P_{mk}A_k^2\bar{A}_k\right)\exp(i\omega_k T_0) \\ &+\frac{1}{9\omega_k^2-\omega_m^2}\sum_{k=1}^{n}P_{mk}A_k^3\exp(3i\omega_k T_0) \\ &+\frac{1}{2(\omega_m^2-\omega_1^2)}G_m\exp(i\sigma T_1)\exp(i\omega_1 T_0)+cc\ (k\neq m) \end{aligned} \tag{22}$$

By substituting Eq. (19) and Eq. (22) into the third equation of Eq. (17), the equation can be solved. However, it is quite complex to solve all equations for $m=2, 3, \sim 2n$ component equation. Thus, the equation is arranged according to the condition to eliminate the secular term (the terms of $i\omega_m T_0$) as follows:

$$\begin{aligned} &-2i\omega_m D_2 A_m+Q_{mm}^2 A_m+\frac{3}{\omega_m}iQ_{mm}P_{mm}A_m^2\bar{A}_m \\ &+\frac{9}{4\omega_m^2}P_{mm}^2 A_m^3\bar{A}_m^2+cc=0 \end{aligned} \tag{23}$$

When the vibration is steady state, by considering Eq. (23), the developed equation can be obtained. Then, the arranged equation can be rewritten in terms of the real part and the imaginary part.

In accordance with formulation to the perturbation first order, this result corresponds to

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the $\eta_2$, $\eta_3$, $\sim\eta_{2n}=0$ relation when the nonlinear restoring force is transformed into modal coordinates. The particular solution of the single DOF $\eta_1^{(1)}$ of Eq. (21) becomes

$$\eta_1^{(1)}=\frac{1}{8\omega_1^2}P_{11}A_1^3\exp(3i\omega_1 T_0)+cc \tag{24}$$

By substituting Eq. (19), and Eq. (24) into the third equation in Eq. (17), the equation can be arranged in a simple form. Using secular term (the terms of $i\omega_1 T_0$), the single DOF equation can be obtained as follow:

$$\begin{aligned}&-2i\omega_1 D_2 A_1+Q_{11}^2 A_1+\frac{3}{\omega_1}iP_{11}Q_{11}A_1^2\bar{A}_1+\frac{9}{4\omega_1^2}P_{11}^2A_1^3\bar{A}_1^2\\&+\frac{G_1}{4\omega_1}\left(iQ_{11}-\frac{3}{\omega_1}P_{11}A_1\bar{A}_1-\sigma\right)\exp(i\sigma T_1)\\&+\frac{3}{8\omega_1^2}P_{11}G_1A_1^2\exp(-i\sigma T_1)+cc=0\end{aligned} \tag{25}$$

where $A_1=\frac{1}{2}a_1\exp(i\beta_1)$. By substituting $A_1$ into Eq. (25), the equation can be rewritten in terms of the real part and the imaginary part. When the vibration is steady state and the relation of $\left(\frac{dA}{dt}=\varepsilon D_1 A+\varepsilon^2 D_2 A=0\right)$ is considered, the equation becomes

$$A_C\cos\gamma+B_C\sin\gamma=C_C,\ D_C\cos\gamma+E_C\sin\gamma=F_C \tag{26}$$

$$A_C=\left(\frac{1}{2}-\frac{\varepsilon\sigma}{4\omega_1}-\frac{3\varepsilon P_{11}}{32\omega_1^2}a_1^2\right)G_1,$$

$$B_C=-\frac{G_1\varepsilon}{4\omega_1}Q_{11},\ D_C=-B_C$$

where $C_C=a_1\left(-\frac{\varepsilon}{2}Q_{11}^2-\omega_1\sigma+\frac{3}{8}P_{11}a_1^2-\frac{9\varepsilon}{128\omega_1^2}P_{11}^2a_1^4\right)$

$$E_C=\left(\frac{1}{2}-\frac{\varepsilon\sigma}{4\omega_1}-\frac{9\varepsilon P_{11}}{32\omega_1^2}a_1^2\right)G_1,$$

$$F_C=\omega_1 Q_{11}a_1-\frac{3\varepsilon}{8\omega_1}Q_{11}P_{11}a_1^3$$

By eliminating the term $\gamma$ from the Eq. (26), the polynomial equation of $a^2$ is obtained as follows;

$$\sum_{n=0}^{7}C_n(\varepsilon,\ \sigma,\ \omega_1,\ P_{11},\ Q_{11},\ G_1)\,a^{2n}=0 \tag{27}$$

The frequency response of the system to the perturbation second order is obtained by solving Eq. (27). From the Eq. (27), the solution of the equation of motion to the first order of $\varepsilon$ is obtained. The response for the single DOF can be expressed as

$$\eta_1=a_1\cos(\Omega t-\gamma)+\varepsilon\left\{\frac{1}{32\omega_1^2}P_{11}a_1^3\cos(3\Omega t-3\gamma)\right\} \tag{28}$$

where $\gamma$ is obtained from the Eq. (26). The time response of the equation of motion can be obtained by changing Eq. (28) into physical coordinates.

## 4. Numerical Examples

Here, the response analysis is presented to demonstrate the application of the proposed method. The responses of the proposed method are compared with those obtained by the classical analysis technique for accuracy validation.

A nonlinear rotor system, which is shown in Fig. 3, is considered. The rotor and the casing are considered to be a uniform beam approximately for the simplicity of calculation. The cross-coupling terms in the bearing are ignored for the simplicity to verify the effect of nonlinearity. The properties of the rotor system are tabulated in Table 1.

**Table 1** Properties of the rotor system

| | | |
|---|---|---|
| Rotor, Casing length | (mm) | $1.6\times10^3$ |
| Rotor diameter | (mm) | $3.0\times10^2$ |
| Casing diameter | (mm) | $1.0\times10^2$ |
| Young's modules of rotor and casing | (N/m²) | $2.1\times10^{11}$ |
| Density of rotor and casing | (Kg/m³) | $7.81\times10^4$ |
| Bearing coefficient | (N/m) | $6.69\times10^4$ |
| Constrain coefficient | (N/m) | $1.0\times10^{10}$ |

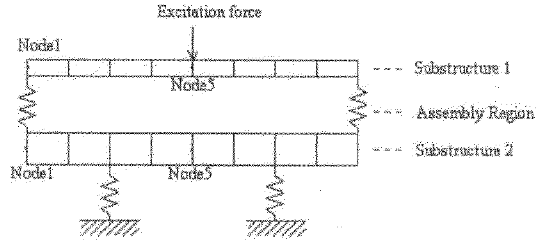


**Fig. 3** Model for Analysis

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The rotor and the casing are modeled with the eight beam elements. The modal damping ratios of the rotor system are $\alpha$, $\beta=0.05$. The external unbalance force with as value of 50 is acted on the 5th nodal point of substructure 1. The perturbation parameter for the nonlinearity is adopted as a small value (that is, $0\leq\varepsilon\leq0.6$). If the bigger value for $\varepsilon>0.6$ is adopted, the solution will be deviate from the exact one. This is one of the limitation of the multiple scales method.

Figure 4 shows the frequency responses which are calculated by perturbation first order approximation at the nodal point 5 of substructure 1 when adopting 5 modes and total modes (18 modes).

Figure 5 shows the frequency responses which are calculated by perturbation second order approximation at the nodal point 5 of substructure 1 when adopting 5 modes and total modes (18 modes).

It can be observed that the presented method show relatively accurate frequency responses by adopting 5 modes compared with responses of adopting all modes. From this result, it is believed that the nonlinear restoring force term can be easily transformed into modal coordinates while retaining its accuracy with its lower modes according to the proposed procedure. To evaluate the proposed technique, the responses need to be compared with the other representative nonlinear analyzing method, such as direct integration method. Using the FEM, the equation of motion of rotor system, which is composed of rotor-bearing-casing, is obtained. Numerical integration is carried out conveniently in terms of first order equation. Thus, the nonlinear equation is recast in the state form. Then, the fourth order Runge-kutta method is used to obtain the response for unbalance excitation.

Figures 6 and 7 show the frequency responses, determined by using the MS method with the first order and the second order. Those responses are obtained at the nodal point 5 of substructure 1. Five modes are adopted in each response.

As shown in the reference (Hassan, 1994), "Incorrect solutions," which do not exist in the direct numerical integration response, appears to be the solution when using the MS method with approximating to the second order. Though "Incorrect solutions" appear in the large amplitude area around 0.03 or more of the frequency

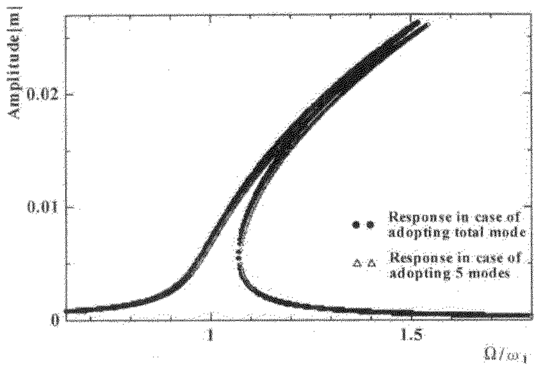


**Fig. 4** Frequency response by using first order approximation

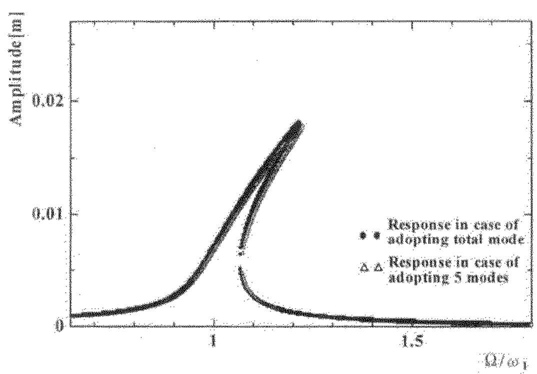


**Fig. 5** Frequency response by using second order approximation

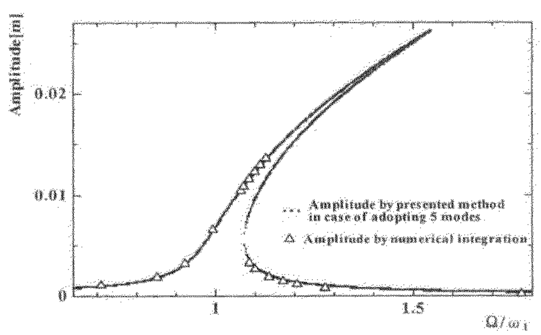


**Fig. 6** Comparison of Frequency response by using first order approximation

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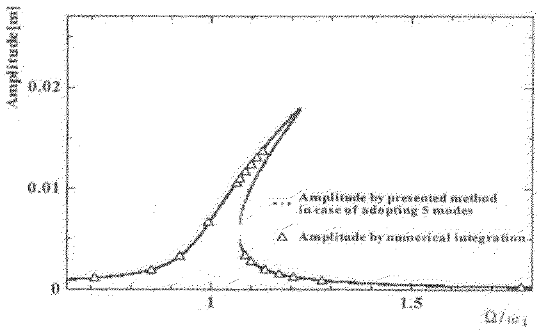


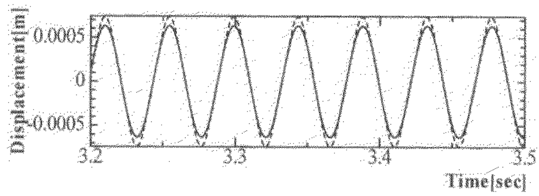

**Fig. 7** Comparison of Frequency response by using second order approximation

response curve of Figs. 5 and 7, they are neglected because it is unrelated in substance with this study so it is not shown on the graph. It can be observed that the results of each method showed well the nonlinear characteristic in comparatively good agreement with the those of direct integration, as shown in Figs. 6 and 7. Especially, there is a good agreement with keeping the accuracy of the response between the proposed method of the perturbation second order and integration method as shown in Fig. 7.

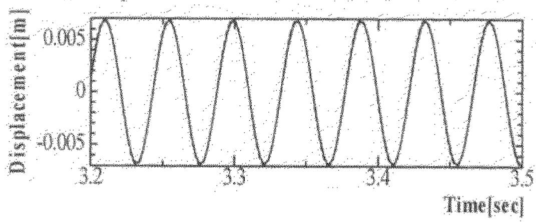
As a result, the proposed method can be employed for the frequency response with relatively compact formulation of the complex system with the almost same degree of accuracy with the direct numerical integration. In this study, the steady state response is analyzed but the stability distinction is not carried out.

Figures 8 and 9 show the system response in the time domain response results using the MS method adopting 5 modes in accordance with Eq. (40) and the direct integration method at node 1 and node 5 of substructure 1. Figure 8 compares two time domain responses when the system is excited by an external force with exciting frequency of 138rad/s where the first natural frequency of the system is 141rad/sec.

Figure 9 compares two time domain responses when the structure is excited by external force with exciting frequency 157rad/s, which is a little larger than the first natural frequency of the system. Compared with the amplitude of the response by direct integration method, it can be observed at the selected point that comparatively




(a) Displacement at Node 1 in substructure 1

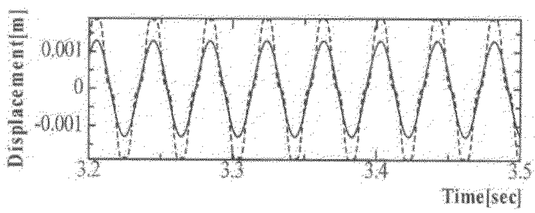



(b) Displacement at Node 5 in substructure 1

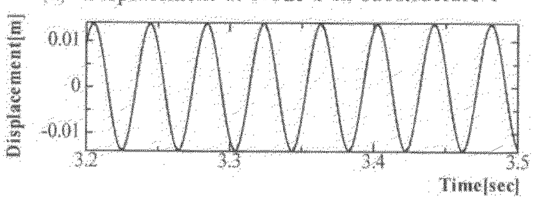
——— Presented method,

------- Integration method

**Fig. 8** Comparison of time history. ($\Omega$=138rad/sec)




(a) Displacement at Node 1 in substructure 1




(b) Displacement at Node 5 in substructure 1

——— Presented method,

------- Integration method

**Fig. 9** Comparison of time history. ($\Omega$=157rad/sec)

accurate nonlinear responses of the system are illustrated with the corresponding phase. Nevertheless, there is a little difference of responses at node 1 of substructure 1, as shown in Figs. 8 and 9. By careful examination of the responses both at nodal point 1 and nodal point 5 in Figs. 8 and 9, it is reveals that the nonlinear displacement becomes small though it is excited near the first natural frequency. This can be understood that

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the amplitude of the frequency response grows up even if the exciting frequency exceeds the first natural frequency because of the effect of nonlinearity, as observed in Figs. 6 and 7.

Figure 10 shows the corresponding FFT (Fast Fourier Transform) analysis results of time domain response at node 1 and node 5 of substructure 1, which are calculated by the direct numerical integration and the MS method. Each time responses are obtained by the same simulation condition with Fig. 9. The power spectrum is expressed in a logarithm to confirm the nonlinear frequency element easily in the diagram. Both results in Fig. 10, are comparatively in a good agreement.

The nonlinear frequency element (3Ω) is observed in each spectrum. Nevertheless, it is observed that the spectrum of nonlinear frequency element (3Ω) of the proposed method is smaller than the spectrum obtained by the direct numerical integration method at node 1 of substructure 1. It is assumed that the response of MS method is approximated response. Accordingly, there might be an increase in deviation from the exact solution. There is no higher nonlinear frequency element (5Ω) in the presented method while the result of the integration method shows one. Because the proposed method approximated the

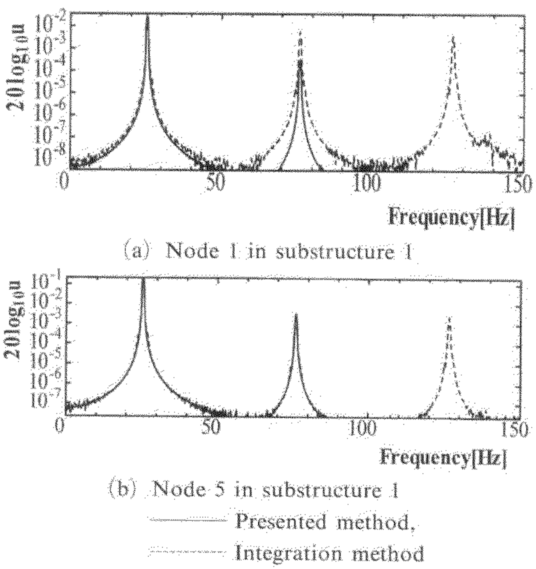


**Fig. 10** Comparison of frequency spectra

solution to the frequency (3Ω) element, there is no frequency element (5Ω). Next, the calculation time is considered to verify the effectiveness of the proposed method. For example, the calculation time for the responses in Fig. 9 is examined. The proposed method takes 2 minutes 45 seconds to calculate the time response until the 3.5 second time interval, while the direct integration method takes 18 minutes 35 seconds to compute the same time interval by using the personal computer 의 Logix IBM Co.. As a result, it can be observed in this study that a drastic reduction in computational time can be achieved with retaining the accuracy. This is a critical factor in the analysis of the structural dynamics with a large number of DOF systems.

## 5. Conclusions

In this paper, the vibration analysis of a nonlinear mechanical system has been formulated theoretically by employing the MS method. The formulation is concerned with reducing the number of DOF for each substructure by modal substitution in accordance with the MS theory. All the substructures are then re-assembled together and the nonlinear response of the overall system is obtained for the harmonic excitation. This method was applied to a nonlinear rotor system. The performance of the proposed method was compared with the direct integral method in terms of the computational accuracy and time. It has been shown that the nonlinear responses can be efficiently calculated with the selected number of vibration modes. And the nonlinear characteristic of the nonlinear restoring force is well simulated. As a result, the proposed method was proved to be an applicable technique for analyzing the dynamics of the nonlinear structures. Moreover, it is believed that those properties of the results can be utilized in the dynamic design of the nonlinear system.

## Acknowledgment

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